Extended Period Simulation of Water Systems—Direct Solution

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Abstract: Extended period simulation or dynamic analysis of water distribution systems helps in their proper operation by checking whether the flow rates are maintained at adequate pressures at all nodes and whether the storage properly balances the supply and distribution. A direct procedure is developed and illustrated in this paper for such dynamic analysis. The procedure can be applied to formulate Q-, ΔQ-, or H-equations and therefore dynamic analysis can be directly obtained by the usual linear theory, or by the Newton-Raphson or Hardy Cross methods of static network analysis. The available computer programs based on any of these methods can be used, with minor modifications, for carrying out dynamic analysis of water-distribution systems.

Introduction

A common practice of analyzing flow in water-distribution systems is to assume the flow to be in a steady-state condition. This is "static analysis." However, neither the nodal demands nor the reservoir water levels remain constant over a period of time. To ensure an adequate level of service to the consumers under varying conditions of demands and reservoir water levels, proper operation of the distribution system is necessary. From an operational point of view, it is necessary to adequately maintain the flow rates and pressures (residual heads) at all nodes at various times; it is also necessary to manage the storage to balance the supply and distribution. These objectives can be achieved by carrying out the analysis of the network over a period of 24–48 hr under varying nodal demands and reservoir water levels. Such an analysis of the distribution system is an extended period simulation or simply a "dynamic analysis."

The necessity of dynamic analysis for water-distribution systems was recognized by some investigators including Shamir and Howard (1968) and Tart (1973). However, Rao and Bree (1977) and Rao et al. (1977), in their classic papers, were the first to suggest a systematic procedure for carrying out dynamic analysis of water-distribution systems. Even though their method is iterative, it is easy to understand and is directly based on static analysis. Their technique integrates several static solutions, each at the end of a preselected time interval, by considering the changes in reservoir water levels due to fill up and depletion, changes in pumping schedules, the effect of boundary elements bringing water into the network from a source outside the network, and the changes in nodal demands. The integration procedure is based on a differential equation for the reservoir heads as a function of time. This differential equation is used to update the static

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solution at the end of the preselected time interval using an iterative predictor-corrector procedure that is continued until the heads predicted at the beginning of the iterative procedure match with those obtained at the end.

A direct procedure is developed and illustrated in this paper for carrying out dynamic analysis of water-distribution systems. The proposed procedure can be applied to formulate Q-equations, ΔQ-equations, or H-equations and therefore dynamic analysis can be obtained by the linear theory, the Newton-Raphson method, or the Hardy Cross method of network analysis. The available computer programs based on any of these static-analysis methods need minor modifications so that they can also be used for carrying out dynamic analysis of water-distribution systems.

Static Analysis

The static analysis of pipe networks is based on the following relationships.

Pipe Head-Loss Relationship

The Hazen-Williams, Darcy-Weisbach, and Manning head-loss relationships are commonly used to express the head loss in a pipe. These relationships can be expressed by a general head-loss relationship.

$$h_{L_x} = H_i - H_j = R_x Q_x^n \quad (1)$$

in which h_{L_x} = head loss in pipe x, L; H_i, H_j = heads at upstream node i and downstream node j of pipe x, respectively, L; R_x = resistance of pipe x; Q_x = discharge in pipe x, L^3/T; and n = exponent that normally lies between 1.7 and 2.0.

Eq. 1 can also be expressed as

$$h_{L_x} = H_i - H_j = R_x |Q_x|^{n-1} Q_x \quad (2)$$

to avoid raising a negative value when the flow direction changes.

Using Eq. 1, the pipe flow can be expressed as

$$Q_x = \left(\frac{H_i - H_j}{R_x}\right)^{1/n} \quad (3)$$

which is also expressed as

$$Q_x = \left[\frac{|H_i - H_j|^{(1/n)-1}}{R_x^{1/n}}\right](H_i - H_j) \quad (4)$$

to avoid raising a negative value when $H_i < H_j$.

Node Flow Continuity Relationship

For steady incompressible flow in a pipe network, the algebraic sum of the flows at a node must be zero. Thus,

$$\sum_{\substack{x \text{ connected} \\ \text{to } j}} Q_x + q_j = 0, \qquad \text{for all } j \quad (5)$$

in which q_j = external flow (inflow or outflow) at node j, L^3/T.

Loop Head-Loss Relationship

The algebraic sum of the head losses in pipes forming a loop must be zero. Thus,

$$\sum_{x \in c} h_{L_x} = 0, \quad \text{for all } c \quad (6)$$

in which c denotes circuit (loop) in the network. Eq. 6 can be generalized to include minor head losses, and heads supplied by pumps, if any, in the loops (Ormsby and Wood 1986).

The relationships given by Eqs. 1–6 are used to formulate either the Q-equations, ΔQ-equations, or H-equations (Jeppson 1977). In formulating the Q-equations, the pipe discharges (Q_x) are treated as the basic unknowns. These equations are solved to obtain Q_x values, which are subsequently used to obtain other unknown parameters. In formulating ΔQ-equations, the pipe discharges Q_x are assumed initially so that they satisfy the node flow continuity relationship (Eq. 5) at each node and the assumed values are successively corrected by applying a correction ΔQ for each loop so that the final Q_x values also satisfy the loop head-loss relationship (Eq. 6) for each loop. The H-equations are formulated by considering the nodal heads H as the basic unknowns.

The methods commonly used for solving these equations and obtain static analysis are: the linear-theory method (Wood and Charles 1972); the Newton-Raphson method (Martin and Peters (1963); and the Hardy Cross method (Cross 1936; Cornish 1939). The linear-theory method uses the Q-equations after linearizing them if they are nonlinear; the Newton-Raphson method uses either the ΔQ-equations or the H-equations and simultaneously solves them; and the Hardy Cross method uses either the ΔQ-equations or the H-equations but solves them individually after disregarding the effect of the adjacent loops or nodes.

Iterative Procedure for Dynamic Analysis

For the dynamic analysis of a pipe network, the period of analysis is subdivided into several time intervals and the static solution at the end of an interval is linked to that at the beginning of the interval through an iterative integration procedure, suggested by Rao and Bree (1977) and Rao et al. (1977).

A static solution is obtained for time t. This yields the reservoir flows at time t. Using these reservoir flows and the known reservoir water levels at time t, and the integrated value of the total network demand in the time interval Δt, between times t and $t + \Delta t$, reservoir water volumes are predicted for time $t + \Delta t$. The reservoir water elevations at time $t + \Delta t$ are calculated using the capacity-elevation curves for the reservoirs. These predicted reservoir water elevations are used to obtain static solution at time $t + \Delta t$. Since the flow rates at the reservoirs, as determined by the network balance for time $t + \Delta t$, will differ from those used in the predictor calculations, corrector calculations are carried out to predict reservoir volumes and water elevations. The iterative procedure is repeated until the predicted and corrected reservoir water elevations for time $t + \Delta t$ agree to the desired degree of accuracy. Thus, the static solution for time t is linked to the static solution for time $t + \Delta t$ through an iterative, predictor-corrector integration step, taking into account the changes in reservoir water elevations due to fill-up and depletion, the schedule of the pump and valve settings and the variations in the nodal demands. The updated static solution for time $t + \Delta t$ becomes input for the next time interval.

As the solution at each time is static, any one of the network analysis techniques described earlier can be used.

The iterative dynamic analysis procedure consists of the following steps:

1. At time t, the following data are available:
   a. Reservoir water elevations $H_r(t)$ and reservoir water volumes $V_r(t)$ for all r, $r \in M$, the set of reservoirs.
   b. Nodal demands $q_j(t)$ for all j, $j \in N$, the set of demand nodes.
   c. If the network contains boundary elements such as pumps that input water into the network from external sources or from the neighboring pressure zones, then the relationship between the flows $q_b(t)$ and the heads and therefore water elevations with reference to a fixed datum, $H_b(t)$ at the boundary elements b, $b \in B$, the set of boundary elements must be known. The flows and water elevations at the boundary elements may be a function of network demand and of the water purchase contracts.

2. Using the data in step 1, static analysis of the network is obtained for time t. After this solution the heads and flows at all nodes are known.

3. Assuming the outflow rate $q_r(t)$ for reservoir r, as obtained in step 2, to be constant in the time interval $(t, t + \Delta t)$, the depletion in reservoir water volume is obtained as

$$\Delta V_r(t, t + \Delta t) = q_r(t)\, \Delta t \quad \text{(7)}$$

4. The net outflow from all the reservoirs is computed as

$$\sum \Delta V_r = \sum_{r \in M} q_r(t)\, \Delta t \quad \text{(8)}$$

Similarly, the net flow from all the boundary elements is computed as

$$\sum \Delta V_b = \sum_{b \in B} q_b(t)\, \Delta t \quad \text{(9)}$$

5. By plotting a curve of demand versus time for the entire region under consideration, the integrated total demand in the time interval $(t, t + \Delta t)$, i.e., $D(t, t + \Delta t)$ is obtained from the area under the curve for time interval $(t, t + \Delta t)$.

6. The error in water volume balance for the network is predicted as

$$E_p(t, t + \Delta t) = \sum_{r \in M} q_r(t)\, \Delta t + \sum_{b \in B} q_b(t)\, \Delta t + D(t, t + \Delta t) \quad \text{(10)}$$

in which E_p = predicted error for the entire network, L^3. Assuming that this error is shared by the reservoirs only and not also by the boundary elements (Rao and Bree 1977), the error is distributed to all reservoirs in proportion to the withdrawal rate at each reservoir. Thus, for reservoir r,

$$e_{rp}(t, t+\Delta t) = \frac{q_r(t)}{\sum_{r \in M} q_r(t)} \times E_p(t, t+\Delta t) \quad \text{(11)}$$

in which e_{rp} = predicted error for reservoir r, L^3.

7. The reservoir volume at time $t + \Delta t$ is predicted as

$$V_{rp}(t+\Delta t) = V_r(t) + q_r(t)\,\Delta t + e_{rp}(t, t+\Delta t) \quad \text{(12)}$$

8. Using the known capacity-elevation relationship for reservoir r, i.e., $V_r = f_r(H_r)$, the predicted volume $V_{rp}(t + \Delta t)$ yields the predicted water elevation $H_{rp}(t + \Delta t)$ for reservoir r at time $t + \Delta t$.

9. Using the $H_{rp}(t + \Delta t)$ $(r \in M)$, $q_j(t + \Delta t)$ $(j \in N)$, and the relevant data for the boundary elements, static analysis is performed for time $t + \Delta t$.

10. The system is checked for preset switch points for controlling valves and pumps. If one of these is switched in the time interval $(t, t + \Delta t)$, then the next step in the integration procedure is step 13; otherwise, the error in volume balance is recomputed using the new flow rates. Thus,

$$E_c(t, t+\Delta t) = \sum_{r \in M} [q_r(t) + q_r(t+\Delta t)]\frac{\Delta t}{2}$$

$$+ \sum_{b \in B} [q_b(t) + q_b(t+\Delta t)]\frac{\Delta t}{2} + D(t, t+\Delta t) \quad \text{(13)}$$

in which E_c = corrected error for the entire network, L^3. This error is reallocated to reservoir r, in proportion to the average flow rate. Thus,

$$e_{rc}(t, t+\Delta t) = \frac{q_r(t) + q_r(t+\Delta t)}{\sum_{r \in M} q_r(t) + \sum_{r \in M} q_r(t+\Delta t)} \times E_c(t, t+\Delta t) \quad \text{(14)}$$

11. The reservoir volumes are corrected using the corrector equation

$$V_{rc}(t+\Delta t) = V_r(t) + [q_r(t) + q_r(t+\Delta t)]\frac{\Delta t}{2} + e_{rc}(t, t+\Delta t) \quad \text{(15)}$$

With these corrected reservoir volumes, water elevations $H_{rc}(t + \Delta t)$ are recomputed for all r, $r \in M$, from the respective capacity-elevation relationships $V_r = f_r(H_r)$, $r \in M$.

12. The difference between the predicted and corrected water elevations is estimated for all r, $r \in M$. If the difference is more than the permissible limit, the predictor-corrector integration step is repeated for the same time interval. If the difference is within the permissible limt for all reservoirs, the dynamic analysis for time interval $(t, t + \Delta t)$ is complete. The finally obtained corrected values $H_{rc}(t + \Delta t)$ and $V_{rc}(t + \Delta t)$ are set to $H_r(t + \Delta t)$ and $V_r(t + \Delta t)$ values, respectively, for all reservoirs. These values then serve as the starting values for the next time interval.

13. If there is a switch of a pump or a valve in the time interval $(t, t + \Delta t)$, then the time $t + \Delta' t$ $(t < t + \Delta' t < t + \Delta t)$ at which the switching occurs is determined. The time interval is reduced from Δt to $\Delta' t$ and the integration procedure is carried out for two time intervals $(t, t + \Delta' t)$, and $(t + \Delta' t, t + \Delta t)$.

The entire procedure is cycled through the various time intervals to cover the entire period of analysis to complete the dynamic analysis of the network.

Proposed Direct Procedure

In the iterative procedure described earlier, $q_r(t + \Delta t)$ is taken as $q_r(t)$ for the initial predictor-corrector integration iteration (steps 3–8) and is successively corrected (steps 9–12) so that the predicted $H_r(t + \Delta t)$ and $q_r(t + \Delta t)$ values agree with the corresponding corrected values. However, it is possible to obtain a relationship between $H_r(t + \Delta t)$ and $q_r(t + \Delta t)$ and use it in the analysis so that static analysis for time $t + \Delta t$ can be directly obtained from static analysis for time t, instead of adopting the predictor-corrector iterative procedure.

From the known capacity-elevation relationship $V_r = f_r(H_r)$ for reservoir r, $\Delta V_r = f_r'(H_r)\, \Delta H_r$, in which the prime denotes the first derivative. Thus, outflow from reservoir r in time interval $(t, t + \Delta t)$ is given by

$$\Delta V_r(t, t + \Delta t) = f_r'[H_r(t)][H_r(t + \Delta t) - H_r(t)] \quad (16)$$

which on rearrangement can be expressed as

$$H_r(t + \Delta t) = H_r(t) + \frac{\Delta V_r(t, t + \Delta t)}{f_r'[H_r(t)]} \quad (17)$$

Now $\Delta V_r(t, t + \Delta t)$ is the volume outflow from reservoir r in time interval $(t, t + \Delta t)$ and is equal to the average outflow rate $[q_r(t) + q_r(t + \Delta t)]/2$, multiplied by the time interval Δt, plus the share of the error in the water volume balance (Eq. 15).

From static analysis for time t, the total rate of supply to the network from all reservoirs and boundary elements at time t is $\sum q_r(t) + \sum_b q_b(t)$, and it must be equal to the total rate of nodal demands $\sum_j q_j(t)$. Similarly, this is true for time $t + \Delta t$. Thus, based on the average rate of flow, for time interval $(t, t + \Delta t)$,

$$\left[\sum_r q_r(t) + \sum_r q_r(t + \Delta t)\right]\frac{\Delta t}{2} + \left[\sum_b q_b(t) + \sum_b q_b(t + \Delta t)\right]\frac{\Delta t}{2}$$

$$= \left[\sum_j q_j(t) + \sum_j q_j(t + \Delta t)\right]\frac{\Delta t}{2} \quad (18)$$

However, from the network demand curve, the integrated total network demand and thus the total volume of water leaving the system during the time interval $(t, t + \Delta t)$ is observed to be $D(t, t + \Delta t)$ instead of $[\sum_j q_j(t) + \sum_j q_j(t + \Delta t)]\Delta t/2$. This difference between the observed and estimated total network demand is distributed to the reservoirs and boundary elements in proportion to their withdrawal rates. If $C(t, t + \Delta t)$ is the correction factor, then it is given by

$$C(t, t + \Delta t) = \frac{D(t, t + \Delta t)}{\left[\sum_j q_j(t) + \sum_j q_j(t + \Delta t)\right]\frac{\Delta t}{2}} \quad (19)$$

Therefore, the volume supply from reservoir r in time interval $(t, t + \Delta t)$ is given by

$$\Delta V_r(t, t + \Delta t) = C(t, t + \Delta t)[q_r(t) + q_r(t + \Delta t)] \frac{\Delta t}{2} \qquad (20)$$

Substituting the value of $\Delta V_r(t, t + \Delta t)$ from Eq. 20 in Eq. 17 results in the following:

$$H_r(t + \Delta t) = H_r(t) + \frac{C(t, t + \Delta t)[q_r(t) + q_r(t + \Delta t)] \frac{\Delta t}{2}}{f_r'[H_r(t)]} \qquad (21)$$

which on rearrangement of terms can also be expressed as

$$q_r(t + \Delta t) = \frac{[H_r(t + \Delta t) - H_r(t)]f_r'[H_r(t)]}{C(t, t + \Delta t) \frac{\Delta t}{2}} - q_r(t) \qquad (22)$$

The term $f_r'[H_r(t)]$ represents the cross-sectional area of reservoir r at elevation $H_r(t)$. For reservoirs having constant cross-sectional area $f_r'[H_r(t)]$ remains constant throughout the dynamic analysis; otherwise, it is computed for each time interval. The correction factor $C(t, t + \Delta t)$ is constant for all reservoirs and boundary elements for a particular time interval $(t, t + \Delta t)$.

For a boundary element such as a pump, direct relationship between flow rate and head can be obtained by regression analysis. Thus, $H_b(t) = F_b[q_b(t)]$, or $q_b(t) = f_b[H_b(t)]$, are known. Therefore, for time $t + \Delta t$,

$$H_b(t + \Delta t) = F_b[q_b(t + \Delta t)] \qquad (23)$$

and

$$q_b(t + \Delta t) = f_b[H_b(t + \Delta t)] \qquad (24)$$

The volume supply during time interval $(t, t + \Delta t)$ is given by

$$\Delta V_b(t, t + \Delta t) = C(t, t + \Delta t)[q_b(t) + q_b(t + \Delta t)] \frac{\Delta t}{2} \qquad (25)$$

If the supply rate from a boundary element remains constant during time interval $(t, t + \Delta t)$, i.e., it is $q_b(t)$ throughout the time interval $(t, t + \Delta t)$, correction must not be applied to it. Therefore, while evaluating the correction factor given by Eq. 19, this known and fixed volume $q_b(t)\Delta t$ is subtracted from the numerator as well as the denominator of the right-hand side of Eq. 19.

If a reservoir is filled at a constant rate q_{fr} during time interval $t, t + \Delta t$, Eqs. 21 and 22 would modify to, respectively,

$$H_r(t + \Delta t) = H_r(t) + \frac{C(t, t + \Delta t)[q_r(t) + q_r(t + \Delta t)] \frac{\Delta t}{2} + q_{fr}\Delta t}{f_r'[H_r(t)]} \qquad (26)$$

and

$$q_r(t + \Delta t) = \frac{[H_r(t + \Delta t) - H_r(t)] \times f_r'[H_r(t)] - q_{fr}\Delta t}{C(t, t + \Delta t)\dfrac{\Delta t}{2}} - q_r(t) \quad \text{.........} \quad (27)$$

Eqs. 21–27 are used in formulating appropriate equations for carrying out static analysis for time $t + \Delta t$, as a part of dynamic analysis for time interval t, $t + \Delta t$.

The proposed direct procedure for carrying out dynamic analysis is illustrated for a simple network.

Illustrative Example

The network shown in Fig. 1 has nodes 1 and 2 as reservoirs and nodes 3–6 as demand nodes. The resistance constants in the Hazen-Williams head-loss formula $h_{L_x} = R_x Q_x^{1.852}$, $x = 1$–7 are shown underlined in the figure. The dynamic analysis of the network is required from 0 hr to 24 hr, at a time interval of 4 hr. The nodal demands at different times are given in Table 1. The observed network demands, as obtained from the demand curve, between different time intervals are: 0–4 hr, 1,488 m^3; 4–8 hr, 2,256 m^3; 8–12 hr, 2,304 m^3; 12–16 hr, 2,184 m^3; 16–20 hr, 1,992 m^3; and 20–24 hr, 1,296 m^3. Reservoirs 1 and 2 have constant cross-sectional areas of 1,666.67 m^2 and 1,250 m^2, respectively. Water levels in reservoirs 1 and 2, at time 0 hr, are 101.200 m and 102.000 m, respectively. Water volumes in reservoirs 1 and 2 at time $t = 0$ hr are 2,000 m^3 and 5,000 m^3, respectively. Reservoir 1 is filled at a rate of 14 m^3/min from 4 hr to 12 hr, while reservoir 2 is filled at a rate of 10 m^3/min from 12 hr to 20 hr. Both reservoirs are floating on the network.

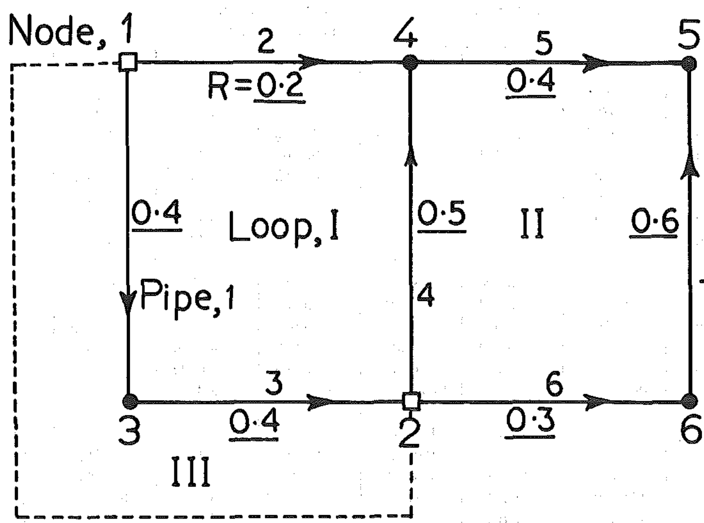

FIG. 1. Illustrative Network

TABLE 1. Demands at Nodes of Illustrative Network

Node (1)	Nodal Demands, in cubic meters per minute, at time t						
	0 hr (2)	4 hr (3)	8 hr (4)	12 hr (5)	16 hr (6)	20 hr (7)	24 hr[a] (8)
3	1.00	2.00	3.00	3.00	2.00	2.00	1.00
4	0.80	1.60	2.00	2.60	1.20	1.60	0.80
5	1.00	3.00	4.00	2.00	4.00	2.00	1.00
6	0.60	1.60	1.00	1.20	2.00	1.40	0.60
Total network demand rate	3.40	8.20	10.00	8.80	9.20	7.00	3.40

[a]Nodal demands are same as those at 0 hr.

Dynamic Analysis for Time Interval 0–4 hr

Static analysis of the network is carried out for $t = 0$ hr and the solution is given in column 2 of Table 2. From Eq. 19, the correction factor is obtained as: $C(0, 4) = 1{,}488/[(3.4 + 8.2) \times 240/2] = 1.06897$.

For reservoir 1, area $f_1'[H_1(0)] = 1{,}666.67\ \text{m}^2$; and for reservoir 2, area $f_2'[H_2(0)] = 1{,}250\ \text{m}^2$. Therefore, for reservoir 1, from Eqs. 21 and 22,

$$H_1(4) = H_1(0) + \frac{C(0, 4)[q_1(0) + q_1(4)]\frac{\Delta t}{2}}{f_1'[H_1(0)]} = 101.200$$

$$+ \frac{1.06897[0.5850 + q_1(4)]}{1666.67} \times 120 \quad \text{(28a)}$$

TABLE 2. Dynamic Analysis Solution

Item (1)	Static Analysis at Time						
	0 hr (2)	4 hr (3)	8 hr (4)	12 hr (5)	16 hr (6)	20 hr (7)	24 hr (8)
Q_1, m^3/min	−0.3854	0.9525	2.5150	3.2257	1.8102	0.7376	0.5009
Q_2, m^3/min	−0.1995	2.1981	3.8316	4.6296	3.1115	1.5682	0.9069
Q_3, m^3/min	−1.3854	−1.0474	−0.4850	0.2260	−0.1896	−1.2626	−0.4993
Q_4, m^3/min	1.2801	1.3983	0.7649	−0.4382	0.9499	1.3306	0.5518
Q_5, m^3/min	0.2800	1.9962	2.5965	1.5917	2.8615	1.2989	0.6582
Q_6, m^3/min	1.3198	2.6036	2.4034	1.6085	3.1385	2.1012	0.9420
Q_7, m^3/min	0.7196	1.0037	1.4034	0.4082	1.1386	0.7014	0.3415
q_1, m^3/min	0.5850[a]	−3.1506	−6.3466	−7.8553	−4.9217	−2.3058	−1.4079
q_2, m^3/min	−3.9853	−5.0494	−3.6534	−0.9444	−4.2781	−4.6943	−1.9931
q_3, m^3/min	1.0000	2.0000	3.0000	3.0000	2.0000	2.0000	1.0000
q_4, m^3/min	0.8000	1.6000	2.0000	2.6000	1.2000	1.6000	0.8000
q_5, m^3/min	1.0000	3.0000	4.0000	2.0000	4.0000	2.0000	1.0000
q_6, m^3/min	0.6000	1.6000	1.0000	1.2000	2.0000	1.4000	0.6000
V_1, m^3	2,000	1,671.7	3,853.3	5,473.3	3,923.3	3,035.0	2,573.3
V_2, m^3	5,000	3,841.2	2,762.5	2,198.8	3,965.0	5,262.5	4,428.8
H_1, m	101.200	101.003	102.312	103.284	102.354	101.821	101.544
H_2, m	102.000	101.073	100.210	99.759	101.172	102.210	101.543
H_3, m	101.268	100.637	100.105	99.784	101.154	101.594	101.433
H_4, m	101.210	100.143	99.906	99.867	100.718	101.361	101.377
H_5, m	101.172	98.704	97.564	98.921	97.914	100.712	101.192
H_6, m	101.499	99.308	98.688	99.036	98.677	101.023	101.275

[a]Reservoir is being filled by the distribution system.

or

$$H_1(4) = 101.200 + 0.076966[q_1(4) + 0.5850] \quad \text{(28b)}$$

and

$$q_1(4) = 12.9928[H_1(4) - 101.200] - 0.5850 \quad \text{(29)}$$

Similarly, for reservoir 2,

$$H_2(4) = 102.000 + 0.10262[q_2(4) - 3.9853] \quad \text{(30)}$$

and

$$q_2(4) = 9.74458[H_2(4) - 102.000] + 3.9853 \quad \text{(31)}$$

As seen from Fig. 1, as reservoir 1 is being depleted for the assumed flow directions in pipes 1 and 2,

$$q_1(4) = -[Q_1(4) + Q_2(4)] \quad \text{(32)}$$

Similarly, for reservoir 2,

$$q_2(4) = Q_3(4) - Q_4(4) - Q_6(4) \quad \text{(33)}$$

Eq. 28*b*–33 are used in formulating appropriate equations for carrying out dynamic analysis for time interval 0–4 hr.

Formulation of Q-Equations

From node flow continuity relationships at nodes 3–6, respectively, the Q-equations are

$$Q_1(4) - Q_3(4) - 2.00 = 0 \quad \text{(34a)}$$

$$Q_2(4) + Q_4(4) - Q_5(4) - 1.60 = 0 \quad \text{(34b)}$$

$$Q_5(4) + Q_7(4) - 3.00 = 0 \quad \text{(34c)}$$

$$Q_6(4) - Q_7(4) - 1.60 = 0 \quad \text{(34d)}$$

Using Eq. 1 instead of Eq. 2 for simplicity and brevity, from loop head-loss relationships for basic loops I and II, respectively,

$$0.2[Q_2(4)]^n - 0.5[Q_4(4)]^n - 0.4[Q_3(4)]^n - 0.4[Q_1(4)]^n = 0 \quad \text{(34e)}$$

$$0.4[Q_5(4)]^n - 0.6[Q_7(4)]^n - 0.3[Q_6(4)]^n + 0.5[Q_4(4)]^n = 0 \quad \text{(34f)}$$

Similarly, from loop head-loss relationship for a pseudoloop, i.e., loop III:

$$0.4[Q_1(4)]^n + 0.4[Q_3(4)]^n + H_2(4) - H_1(4) = 0 \quad \text{(34g)}$$

Substituting the values of $H_2(4)$ and $H_1(4)$ from Eqs. 30 and 28*b*, respectively, and the values of $q_2(4)$ and $q_1(4)$ therein from Eqs. 33 and 32, respectively, Eq. 34*g* on simplification becomes,

$$0.4[Q_1(4)]^n + 0.4[Q_3(4)]^n + 0.10262[Q_3(4) - Q_4(4) - Q_6(4)]$$
$$+ 0.076966[Q_1(4) + Q_2(4)] + 0.34600 = 0 \quad \text{(34h)}$$

Simultaneous solution of Eqs. 34*a–f* and 34*h* by the linear theory method yields the solution as given in column 3 of Table 2. Note that in ordinary static analysis of the network at $t = 4$ hr, the values of $H_1(4)$ and $H_2(4)$ are known and therefore simultaneous solution of Eqs. 34*a–g* would yield the ordinary static analysis solution. In dynamic analysis, the values of $H_1(4)$ and $H_2(4)$ are expressed in terms of the pipe flows. Therefore, for dynamic analysis, Eq. 34*g* is replaced by Eq. 34*h* and thus only the equation for the pseudoloop has changed. The total number of the equations has remained the same, i.e., seven as for the ordinary static analysis.

Formulation of ΔQ-Equations

Assuming the values of $Q_1(4)$–$Q_7(4)$ so that they satisfy Eqs. 34*a–d* denoting these values by suffix 0, omitting 4 within parentheses for simplicity, and taking ΔQ_I–ΔQ_{III} as loop-flow corrections (clockwise positive) for loops I–III, respectively, Eqs. 34*e–h* become, respectively,

$$0.2(Q_{02} + \Delta Q_I)^n - 0.5(Q_{04} - \Delta Q_I + \Delta Q_{II})^n - 0.4(Q_{03} - \Delta Q_I + \Delta Q_{III})^n$$

$$- 0.4(Q_{01} - \Delta Q_I + \Delta Q_{III})^n = 0 \quad \text{(35a)}$$

$$0.4(Q_{05} + \Delta Q_{II})^n - 0.6(Q_{07} - \Delta Q_{II})^n - 0.3(Q_{06} - \Delta Q_{II})^n$$

$$+ 0.5(Q_{04} - \Delta Q_I + \Delta Q_{II})^n = 0 \quad \text{(35b)}$$

$$0.4(Q_{01} - \Delta Q_I + \Delta Q_{III})^n + 0.4(Q_{03} - \Delta Q_I + \Delta Q_{III})^n + H_2(4) - H_1(4) = 0 \quad \text{(35c)}$$

$$0.4(Q_{01} - \Delta Q_I + \Delta Q_{III})^n + 0.4(Q_{03} - \Delta Q_I + \Delta Q_{III})^n + 0.10262$$

$$\cdot [(Q_{03} - \Delta Q_I + \Delta Q_{III}) - (Q_{04} - \Delta Q_I + \Delta Q_{II}) - (Q_{06} - \Delta Q_{II})]$$

$$+ 0.076966[(Q_{01} - \Delta Q_I + \Delta Q_{III}) + (Q_{02} + \Delta Q_I)] + 0.34600 = 0 \quad \text{(35d)}$$

which on simplification becomes

$$0.4(Q_{01} - \Delta Q_I + \Delta Q_{III})^n + 0.4(Q_{03} - \Delta Q_I + \Delta Q_{III})^n + 0.10262$$

$$\cdot (Q_{03} - Q_{04} - Q_{06} + \Delta Q_{III}) + 0.076966(Q_{01} + Q_{02} + \Delta Q_{III}) + 0.34600 = 0 \quad \text{(35e)}$$

Solution of Eqs. 35*a*, 35*b*, and 35*e*, either simultaneously by the Newton-Raphson method or individually by the Hardy Cross method (retaining only ΔQ_I in Eq. 35*a*, ΔQ_{II} in Eq. 35*b*, and ΔQ_{III} in Eq. 35*e*), yields the dynamic analysis solution given in column 3 of Table 2. Note that in the third and fourth terms of Eq. 35*e*, only ΔQ_{III} is present and the correction terms for other loops are absent. Furthermore, the solution of Eqs. 35*a–c* yields the ordinary static solution for $t = 4$ hr, and the number of equations in dynamic analysis is also three, as in ordinary static analysis.

Formulation of H-Equations

H-equations are formulated by writing node flow continuity equations (flows are expressed in terms of nodal heads) at all the six nodes of the network. Using Eq. 3 instead of Eq. 4 and omitting 4 within parentheses for simplicity and brevity, the H-equation for node 1 is

$$-\left(\frac{H_1 - H_3}{0.4}\right)^{1/n} - \left(\frac{H_1 - H_4}{0.2}\right)^{1/n} - 12.9928(H_1 - 101.200) + 0.5850 = 0 \quad \text{(36a)}$$

Similarly, for node 2,

$$\left(\frac{H_3 - H_2}{0.4}\right)^{1/n} - \left(\frac{H_2 - H_4}{0.5}\right)^{1/n} - \left(\frac{H_2 - H_6}{0.3}\right)^{1/n}$$

$$- 9.74458(H_2 - 102.000) - 3.9853 = 0 \quad (36b)$$

For nodes 3–6, respectively,

$$\left(\frac{H_1 - H_3}{0.4}\right)^{1/n} - \left(\frac{H_3 - H_2}{0.4}\right)^{1/n} - 2.00 = 0 \quad (36c)$$

$$\left(\frac{H_1 - H_4}{0.2}\right)^{1/n} - \left(\frac{H_2 - H_4}{0.5}\right)^{1/n} - \left(\frac{H_4 - H_5}{0.4}\right)^{1/n} - 1.60 = 0 \quad (36d)$$

$$\left(\frac{H_4 - H_5}{0.4}\right)^{1/n} + \left(\frac{H_6 - H_5}{0.6}\right)^{1/n} - 3.00 = 0 \quad (36e)$$

$$\left(\frac{H_2 - H_6}{0.3}\right)^{1/n} - \left(\frac{H_6 - H_5}{0.6}\right)^{1/n} - 1.60 = 0 \quad (36f)$$

The solution of Eqs. 36*a–f*, either simultaneously by the Newton-Raphson method or individually by the Hardy Cross method, yields the dynamic analysis for 0–4 hr. Note that for ordinary static analysis at $t = 4$ hr, as the values of $H_1(4)$ and $H_2(4)$ are known, the static analysis is obtained by solution of Eqs. 36*c–f*, while dynamic analysis requires all the six equations. Thus, in dynamic analysis the number of required equations increases by M, the number of reservoirs. (Because of this, depending upon the number of reservoirs relative to the number of other nodes, the computer runtime for dynamic analysis by direct procedure could be more than that by the predictor-corrector iterative procedure.)

Reservoir 1 is filled at the rate of 14 m^3/min between 4 and 12 hr. Therefore, for the dynamic analysis of time intervals 4–8 hr and 8–12 hr,

TABLE 3. Dynamic Analysis by Predictor-Corrector Iterative Procedure for Time Interval 0–4 hr

	Predicted Flow (m^3/min) for $t = 4$ hr		Corrected Head (m) for $t = 4$ hr	
Iteration (1)	Reservoir 1 (2)	Reservoir 2 (3)	Reservoir 1 (4)	Reservoir 2 (5)
Initial (steps 3–8)	+0.5850	−3.9853	100.921	101.181
1 (steps 9–12)	−4.2081	−3.9918	101.021	101.048
2 (steps 9–12)	−2.9072	−5.2928	100.998	101.079
3 (steps 9–12)	−3.2065	−4.9935	101.004	101.072
4 (steps 9–12)	−3.1376	−5.0624	101.002	101.073
5 (steps 9–12)	−3.1534	−5.0465	101.003	101.073

Eqs. 26 and 27 are used instead of Eqs. 21 and 22, respectively, in formulating the Q-, ΔQ-, or H- equations. Similar to this is the case for reservoir 2 between 12 and 20 hr.

The complete dynamic analysis solution is given in Table 2.

The dynamic analysis is carried out, for comparison purposes, by the predictor-corrector iterative integration procedure for time interval 0–4 hr. The results are tabulated in Table 3. As seen from the table, five predictor-corrector iterations are necessary to obtain the results given in column 3 of Table 2.

Summary and Conclusions

A direct procedure is developed and illustrated for carrying out dynamic analysis of water-distribution systems. The application of the procedure for formulation of Q-, ΔQ-, and H-equations is illustrated. In Q- and ΔQ-equations, the number of the unknowns for dynamic analysis remains the same as that in the static analysis, and only the Q- and ΔQ-equations for the pseudoloops change. In the formulation of H-equations, additional H-equations—one for each reservoir—are introduced. The proposed procedure can be used in practice by suitably modifying the available computer programs based on the linear-theory, Newton-Raphson, or Hardy Cross methods of network analysis.

Appendix I. References

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Appendix II. Notation

The following symbols are used in this paper:

B = set of boundary elements;
C = correction factor;

D	=	total network demand as obtained from demand curve;
E	=	error for entire network;
e	=	error at reservoir or boundary element;
F, f	=	function of;
H	=	head at node;
h_L	=	head loss in pipe;
M	=	set of reservoirs;
N	=	set of demand nodes;
n	=	exponent of discharge in pipe head-loss formula;
Q	=	discharge in pipe;
q	=	flow at demand node, reservoir, or boundary element;
R	=	resistance of pipe;
t	=	time;
(t)	=	at time t;
$(t, t + \Delta t)$	=	between time interval t and $t + \Delta t$;
V	=	volume of water in reservoir;
ΔH	=	change in head;
Δt	=	time interval; and
ΔV	=	change in volume of water in reservoir.

Subscripts

b	=	boundary element;
c	=	circuit, corrected;
f	=	filling;
i	=	upstream node;
j	=	downstream node;
p	=	predicted;
r	=	reservoir; and
x	=	pipe.